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Nonlinear convection in a rotating fluid

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The paper studies convection in a horizontal layer of fluid rotating about a vertical axis. The flows at large Rayleigh number R, with a single horizontal wave-number, are investigated using the mean-field approximation of Herring (1963). The flow that maximizes the heat flux is the same as that which gives an upper bound to the heat flux in the limit of infinite Prandtl number as calculated by the methods of Howard (1963) and Chan (1971, 1974).

Rotation is not significant until the Taylor number Ta exceeds $O(R)$. For $O(R) \ll Ta \ll O[(R\log R)^{\frac{4}{3}}]$, it can increase the rate of heat transfer, a phenomenon noted experimentally by Rossby (1969). It does so because an Ekman layer is formed outside the thermal boundary layer, causing a thinning of the thermal layer. The maximum value of the Nusselt number N is approximately $0{\cdot}177R^{\frac{1}{5}}Ta^{\frac{1}{10}}[\log Ta]^{\frac{1}{5}}$. As the Taylor number increases further into the region $O[(R\log R)^{\frac{4}{3}}] \ll Ta \ll O(R^{\frac{3}{2}})$, the maximum value of N drops sharply, and becomes approximately $0{\cdot}029R^{\frac{3}{2}}Ta^{-1}\log(R^{\frac{3}{2}}/Ta)$. Hence, N now decreases with a further increase of Ta and eventually becomes $O(1)$ as $Ta \to O(R^{\frac{3}{2}})$ and the layer becomes stable.

1. Introduction

The specific problem considered in this paper is that of the effect of rotation on convection between two rigid horizontal boundaries at large Rayleigh numbers. It is well known (Chandrasekhar 1961; Niiler & Bisshopp 1965) that a sufficiently large rate of rotation stabilizes the layer. However, the role of rotation cannot always be to inhibit convection, because Rossby (1969) found experimentally that heat transfer is in fact enhanced in some intermediate range of rotation rates. Theoretical and numerical investigations of convection between two free boundaries by Veronis (1968), Van der Borght & Murphy (1973) and Chan (1974) failed to reproduce this phenomenon, though the numerical studies of convection between two rigid boundaries by Somerville (1971) and Somerville & Lipps (1973) did succeed in doing so. So, too, does the theoretical work of this paper.

Thermal convection at Rayleigh numbers greatly in excess of the critical one is strongly nonlinear. No exact mathematical description of it is known, but two types of approximate methods have been used for its study. Either approximations to the governing equations have been made, or else upper bounds to the heat flux have been calculated. There are several possible approaches to deriving approximate equations for the mean flow properties, though many of the resulting sets of equations are similar. One method is that of the mean-field approximation, in which interactions of fluctuating quantities are ignored, and the only nonlinearity considered is that due to the interaction of vertical velocity and temperature with the mean vertical temperature field. It was used by Herring (1963) and several subsequent workers. Elder (1969) gave some physical justification for this "ruthless approximation" on the grounds that it models correctly the major mechanism of heat transfer outside the thermal sublayer, which is the ejection of blobs of sublayer fluid. Roberts (1966) used a similar set of equations for the mean flow, derived using the evolutionary criterion of Glansdorff & Prigogine (1964), which differ only in that a nonlinear convective acceleration term is included. This term vanishes, however, for certain geometrical patterns of the convection cells, such as two-dimensional rolls or rectangular cells, and also in the limit of infinite Prandtl number. Gough, Spiegel & Toomre (1975) derived similar equations by Galerkin methods.

A philosophically different approach is that due to Howard (1963), who showed how an upper bound to the heat flux can be calculated. Certain of the governing equations are replaced by integrals formed from them, and the resulting equations and integrals form constraints. Since the constraints are less restrictive than the full set of governing equations, the maximum heat flux subject to them is a rigorous upper bound to the actual heat flux. Upper bounds calculated in this manner by Howard and subsequent workers are not greatly in excess of measured heat fluxes, and offer some support to a hypothesis proposed by Malkus (1954), that the actual heat transferred is the maximum possible. In principle, the maximizing flow and temperature field need bear no relation to the actual mean flow and temperature field but, as Howard remarks, "if the upper bound on heat transport is not hopelessly too large, it would require an unreasonable amount of self-restraint not to compare *average properties* of the maximizing fields with experimental observations." Such large amounts of self-restraint have not been evident, and indeed maximizing fields have been found to compare reasonably well with observations.

The analysis of this paper uses the exact momentum equation in the limit of infinite Prandtl number, but approximates the heat equation in the manner of the mean-field approximation. The resulting problem is, in the absence of rotation, the same as that discussed by Roberts (1966, appendix by Stewartson), who obtained solutions of it for the limit of large Rayleigh number by boundary-layer methods. The particular instances of these solutions that maximize the heat flux are identical with those obtained by Chan (1971), who sought an upper bound to the heat flux subject to the exact momentum equation in the infinite Prandtl number limit, and an integrated form of the heat equation. Although this approach leads to equations that differ in part from those of the mean-field approximation, the differences are insignificant in the limit of large Rayleigh number. The present work, which includes the effects of rotation, is consequently closely related to that of Chan (1974), who extended his earlier analysis to the rotating case, and also that of Morgan (1973), who worked in terms of a mean-field approximation. Chan's work is restricted to the case of free horizontal boundaries, whereas we consider rigid ones. This difference becomes insignificant at sufficiently high rotation rates, at which stage our results agree with his. Our results disagree with those of Morgan for reasons that will be explained later.

This work is restricted to the case of a single horizontal wavenumber. It is known (Busse 1969, Chan 1971, 1974) that multi-wave solutions are often possible, and allow greater heat fluxes than single-wave solutions. This issue is discussed in § 4, and it is the intention of one of us (N. R.) to extend the present analysis to the multi-wave case. The basic equations used here are given in § 2, and details of their solutions are in § 3 and an appendix.

2. Governing equations

Following previous workers, we consider a horizontally infinite layer of fluid of depth d, bounded above and below. The upper and lower surfaces are maintained at temperatures T_0 and $T_0+\Delta T$, respectively. The fluid is rotating about the vertical with angular velocity Ω. It is convenient to use non-dimensional variables in which lengths, velocities, time, temperature and pressure are scaled respectively by d, κ/d, d^2/κ, ΔT and $\rho\nu\kappa/d^2$. Here κ is the thermometric conductivity, ρ is the mean density, and ν is the kinematic viscosity. Then, neglecting centrifugal force, and with the usual Boussinesq approximation that density variations are taken into account only in the buoyancy term, the basic equations are

$$\frac{1}{\sigma}\left\{\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\,.\nabla)\,\mathbf{u}\right\}+\frac{2}{E}(\mathbf{k}\times\mathbf{u}) = -\nabla P+RT\mathbf{k}+\nabla^2\mathbf{u}, \tag{2.1}$$

$$\nabla\,.\,\mathbf{u} = 0, \tag{2.2}$$

$$\frac{\partial T^*}{\partial t}+(\mathbf{u}\,.\nabla)\,T^* = \nabla^2 T^*. \tag{2.3}$$

Here $\mathbf{u} = (u, v, w)$ is the velocity vector; T^* is the temperature excess over T_0; T is the deviation of T^* from its horizontal average $\overline{T}^*$; and P is the deviation of the pressure from the hydrostatic value appropriate to $\overline{T}^*$. Also, $\mathbf{k}$ is a unit vector in the vertical direction; $\sigma = \nu/\kappa$ is the Prandtl number; $E = \nu/d^2\Omega$ is the Ekman number; $R = \alpha g d^3\Delta T/\kappa\nu$ is the Rayleigh number, α being the coefficient of volume expansion, and g the acceleration due to gravity.

We shall discuss only steady solutions. For this case, a relation for the horizontal mean temperature can be obtained by averaging equation (2.3), and integrating with respect to z. It is

$$d\overline{T}^*/dz = \overline{wT}-1-\langle wT\rangle. \tag{2.4}$$

Here and subsequently, bars denote horizontal averages, and angle brackets denote a further vertical averaging over the whole layer. It follows that the Nusselt number N, which measures the ratio of the actual heat transfer to that achieved purely by conduction, is

$$N = 1+\langle wT\rangle. \tag{2.5}$$

When the horizontal average of the heat equation (2.3) is subtracted from it, the result is

$$\nabla^2 T - w\frac{d\overline{T}^*}{dz} = \nabla\,.\,(T\mathbf{u})-\frac{\partial}{\partial z}(\overline{wT}). \tag{2.6}$$

The mean-field approximation involves the neglect of the right-hand side term, which has the form of a deviation of a bilinear fluctuating quantity from its horizontal mean.

Having taken the limit $\sigma \to \infty$ in (2.1), the number of dependent variables in the problem can be reduced by two if we eliminate P, u and v, while introducing the vertical component of vorticity $X = \partial v/\partial x - \partial u/\partial y$. Also, following Chan, we shall rescale our remaining dependent variables so that

$$\left.\begin{array}{l} \theta = \langle wT\rangle^{-\frac{1}{2}}R^{\frac{1}{2}}T, \quad \omega = \langle wT\rangle^{-\frac{1}{2}}R^{-\frac{1}{2}}w, \quad \Theta = \langle wT\rangle^{-1}\overline{T}^{*} = \overline{T}^{*}/(N-1), \\ \qquad\qquad \zeta = \tfrac{1}{2}\langle wT\rangle^{-\frac{1}{2}}R^{-\frac{1}{2}}EX. \end{array}\right\} \tag{2.7}$$

The governing differential equations are now

$$\nabla^4\omega + \nabla_1^2\theta - Ta\frac{\partial\zeta}{\partial z} = 0, \tag{2.8}$$

$$\nabla^2\zeta + \frac{\partial\omega}{\partial z} = 0, \tag{2.9}$$

$$\frac{d\Theta}{dz} = \overline{\omega\theta} - 1 - \frac{1}{(N-1)}, \tag{2.10}$$

$$\frac{1}{(N-1)R}\nabla^2\theta = \omega\frac{d\Theta}{dz} = \left[\overline{\omega\theta} - 1 - \frac{1}{(N-1)}\right]\omega. \tag{2.11}$$

Here $\quad \nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \quad$ and $\quad Ta = 4/E^2.$

Ta is the Taylor number; and ω and θ are subject to the integral constraints

$$\langle\omega\theta\rangle = 1. \tag{2.12}$$

Also, from integrating (2.11) with respect to z across the layer,

$$R = R\int_0^1 [1-(N-1)(\overline{\omega\theta}-1)]\,dz = \int_0^1 -\frac{1}{\omega}\nabla^2\theta\,dz. \tag{2.13}$$

Instead of making the mean-field approximation, Chan seeks the maximum value of N subject to the constraints of (2.8) and (2.9), but with (2.11) replaced by the exact integral

$$(N-1) < (1-\overline{\omega\theta})^2 > = 1 - < |\nabla\theta|^2 > /R. \tag{2.14}$$

This integral is obtained by an averaging of T times (2.3) over the whole layer. The Euler equations of the variational analysis lead to

$$\left(\nabla^6 + Ta\frac{\partial^2}{\partial z^2}\right)\left\{\frac{\nabla^2\theta}{(N-1)R} + \left[1 - \overline{\omega\theta} + \frac{\lambda}{(N-1)}\right]\omega\right\} = \nabla^2\nabla_1^2\left\{\left[1 - \overline{\omega\theta} + \frac{\lambda}{(N-1)}\right]\theta\right\}, \tag{2.15}$$

instead of (2.11). Here λ is a constant Lagrange multiplier that can be shown to lie in the range $\frac{1}{2} \leqslant \lambda \leqslant 1$. Now, in all the cases considered in the next section, it can and has been verified, although the details are not given, that either the right-hand side terms of (2.15) are less significant than the left-hand side terms to the lowest order of approximation, or else they yield the identical approximation $\overline{\omega\theta} = 1$ that would be obtained from the left-hand side. Because N is large, the $\lambda/(N-1)$ and $1/(N-1)$ terms are unimportant. Hence solutions that satisfy

(2.11) also satisfy (2.15) to a sufficient degree of approximation; and, when their heat flux is maximized with respect to possible choices of the wavenumber, they give upper bounds to the heat flux under the specified conditions.

Nonlinearity is retained in (2.8)–(2.13) only through the horizontal average $\overline{\omega\theta}$ term. Consequently the horizontal dependence of the unknown functions may be separated out by setting

$$\omega = \sum_n \omega_n(z)\,\phi_n(x,y), \quad \theta = \sum_n \theta_n(z)\,\phi_n(x,y), \quad \zeta = \sum_n \zeta_n(z)\,\phi_n(x,y). \quad (2.16)$$

The functions ϕ_n can be any solution of

$$\nabla_1^2\phi_n(x,y) = -\alpha_n^2\phi_n(x,y), \quad (2.17)$$

for some horizontal wavenumber α_n. Functions with different wavenumbers are naturally orthogonal, and can be chosen to be orthonormal so that

$$\overline{\omega\theta} = \sum_n \omega_n(z)\,\theta_n(z)\,\overline{\phi_n^2} = \sum_n \omega_n(z)\,\theta_n(z). \quad (2.18)$$

We consider only a single wavenumber in the analysis of § 3. This analysis is necessary as a guide to the more general case, as well as being interesting in its own right. With our separation of variables, we now obtain ordinary differential equations that simply involve the single wavenumber α_1 and the z-dependent parts θ_1, ω_1 and ζ_1 of the scaled temperature, vertical velocity and vorticity. The unit subscripts will now be omitted for simplicity, so that

$$\left(\frac{d^2}{dz^2}-\alpha^2\right)^2\omega-\alpha^2\theta = Ta\frac{d\zeta}{dz}, \quad (2.19)$$

$$\left(\frac{d^2}{dz^2}-\alpha^2\right)\zeta+\frac{d\omega}{dz} = 0, \quad (2.20)$$

$$\frac{1}{(N-1)R}\left(\frac{d^2}{dz^2}-\alpha^2\right)\theta = \left[\omega\theta-1-\frac{1}{(N-1)}\right]\omega. \quad (2.21)$$

The $1/(N-1)$ term on the right-hand side of (2.21) can normally be neglected because of the largeness of N. This term is however important in an integrated sense in the constraint

$$R = R\int_0^1[1-(N-1)(\omega\theta-1)]\,dz = \int_0^1\frac{1}{\omega}\left[\alpha^2-\frac{d^2}{dz^2}\right]\theta\,dz, \quad (2.22)$$

because

$$\int_0^1(\omega\theta-1)\,dz = \langle\omega\theta\rangle-1 = 0.$$

This constraint is applied in all our subsequent solutions, either one or other form of the integral, as convenient, being evaluated for the various regions of the flow, to yield an expression for N. In addition, there are the boundary conditions appropriate to rigid surfaces at $z = 0, 1$ that

$$\omega = d\omega/dz = \theta = \zeta = 0. \quad (2.23)$$

The subsequent analysis and solution of (2.19)–(2.22) suppose throughout that both the Rayleigh and Nusselt numbers are large. The magnitude of the Taylor number varies, and different classes of solutions are found for different orders of magnitudes of it. In each case, the principal focus is on the unique solution that maximizes N, although other solutions are given.

3. Single-wavenumber flows between rigid boundaries

3.1. *No rotation*: $Ta = 0$.

The solution for this case is essentially that given by Roberts (1966, appendix by Stewartson) and Chan (1971). The wavenumber α is supposed to be large (which can be justified *a posteriori*), so that the convection cells are narrow. The solutions for ω and θ can be obtained by matching asymptotic approximations in three distinct regions. There is a uniform isothermal interior, in which (2.19) and (2.21) are satisfied by

$$\overline{\omega\theta} = \omega\theta = 1, \quad \omega = \alpha^{-1}, \quad \theta = \alpha. \tag{3.1}$$

Near each surface and adjacent to the interior are intermediate layers of thickness $O(\alpha^{-1})$, in which vertical derivatives are important in the viscous forces, and bring the vertical velocity to its zero boundary value. Defining an appropriate boundary-layer co-ordinate $\xi = \alpha z$ for the lower of these layers (the upper layer is similar), the governing equations are

$$\left(\frac{d^2}{d\xi^2} - 1\right)^2 \alpha\omega = \frac{\theta}{\alpha}, \tag{3.2}$$

from (2.19), and again
$$\omega\theta = 1, \tag{3.3}$$

from (2.21). Hence, $\omega = W(\xi)/\alpha$, where W is the solution of

$$\left(\frac{d^2}{d\xi^2} - 1\right)^2 W = \frac{1}{W}, \tag{3.4}$$

for which $W \to 1$ as $\xi \to \infty$. Although no full analytical solution is available, the form of the solution as $\xi \to 0$ can be found, having imposed the requirements that W and $dW/d\xi \to 0$:

$$W \sim \xi^2 [\log \xi^{-1}]^{\frac{1}{2}}. \tag{3.5}$$

There is one further layer, in which thermal conduction is significant in the heat equation and θ is brought to its zero boundary value. Suppose that this inner layer is of thickness $\epsilon \ll \alpha^{-1}$, and define its appropriate co-ordinate as $\eta = z/\epsilon$. The requirement of matching on to the intermediate layer as $\eta \to \infty$ means that

$$\omega \sim \alpha \epsilon^2 \eta^2 \{\log [(\epsilon\alpha)^{-1}]\}^{\frac{1}{2}} \quad \text{as} \quad \eta \to \infty. \tag{3.6}$$

Hence, appropriately scaled forms of ω and θ are $\hat{\omega}$ and $\hat{\theta}$, where

$$\omega = \epsilon^2 \alpha \hat{\omega} \{\log [(\epsilon\alpha)^{-1}]\}^{\frac{1}{2}}, \quad \hat{\theta} = \epsilon^2 \alpha \theta \{\log [(\epsilon\alpha)^{-1}]\}^{\frac{1}{2}}. \tag{3.7}$$

Equation (2.19) therefore reduces to

$$d^4\hat{\omega}/d\eta^4 = 0, \tag{3.8}$$

and the solution that matches on to the intermediate layer, and that also satisfies the boundary conditions at $\eta = 0$, is $\hat{\omega} = \eta^2$. Equation (2.21) predicts that the thickness of the inner thermal layer must satisfy

$$\epsilon^6\alpha^2 NR \log[(\epsilon\alpha)^{-1}] = 1, \tag{3.9}$$

and that $\hat{\theta}$ must be found as the solution of

$$d^2\hat{\theta}/d\eta^2 = (\hat{\omega}\hat{\theta}-1)\,\hat{\omega} = (\eta^2\hat{\theta}-1)\,\eta^2 \tag{3.10}$$

for which $\hat{\theta} = 0$ at $\eta = 0$ and $\hat{\theta} \to \eta^{-2}$ as $\eta \to \infty$. Specifically,

$$\hat{\theta} = \tfrac{1}{3}\eta \int_0^1 (1-t^2)^{-\frac{1}{3}} \exp(-\eta^3 t/3)\, dt. \tag{3.11}$$

Finally, the integral constraint (2.22) is needed for obtaining a scaling of the solution. Evaluating the rightmost expression, the contribution from the interior dominates that from the intermediate layers and is α^4. Adding the contribution from the two inner thermal layers, we obtain to the lowest order of approximation

$$\left.\begin{aligned} R &\sim \alpha^4 + \frac{2I}{\epsilon^5\alpha^2}\{\log[(\epsilon\alpha)^{-1}]\}^{-1},\\ I &= \int_0^\infty (1-\eta^2\hat{\theta})\, d\eta = (\tfrac{2}{9})^{\frac{1}{3}}[\Gamma(\tfrac{2}{3})]^2 = 1{\cdot}1106. \end{aligned}\right\} \tag{3.12}$$

Equations (3.9) and (3.12) can be solved for ϵ and N in terms of α and R, to give

$$\epsilon \sim (10I)^{\frac{1}{5}}[(R\alpha^2-\alpha^6)\log(R\alpha^{-3}-\alpha)]^{-\frac{1}{5}}, \tag{3.13}$$

$$N \sim \left(\frac{1-\alpha^4/R}{2I}\right)^{\frac{6}{5}}\left[\frac{R\alpha^2}{5}\log(R\alpha^{-3}-\alpha)\right]^{\frac{1}{5}}. \tag{3.14}$$

Then, if N is maximized with respect to α, the maximum value is

$$N \sim \left(\frac{6}{13I}\right)^{\frac{6}{5}}\left(\frac{1}{20}\right)^{\frac{1}{5}}\left(\frac{1}{13}\right)^{\frac{1}{10}} R^{\frac{3}{10}}(\log R)^{\frac{1}{5}},$$

attained when
$$\alpha = (R/13)^{\frac{1}{4}}. \tag{3.15}$$

This result is identical to Chan (1971, (63)), while (3.14) is in agreement with Roberts (1966, appendix by Stewartson (A 28)) as modified for large $a = \alpha$ by the sentences that follow, and with the correction of an obvious misprint. Note that the lowest-order solution for ω is sufficient for calculating N, and that it is not necessary to follow Stewartson and Morgan in calculating an improved approximation.

Although the solution for ζ is uncoupled from that for θ and ω, it is of interest because it becomes coupled when $Ta > 0$ and the vertical component of vorticity then plays an important role. We have $\zeta = 0$ in the interior. In the intermediate layer, ζ must be the solution of

$$\left(\frac{d^2}{d\xi^2}-1\right)\zeta = -\frac{1}{\alpha^2}\frac{dW}{d\xi} \tag{3.16}$$

such that $\zeta \to 0$ as $\xi \to \infty$ and $\xi = 0$ at $\xi = 0$. Hence,

$$\alpha^2\zeta = \tfrac{1}{2}\exp\xi\int_{\xi}^{\infty} W'(\mu)\exp(-\mu)\,d\mu + \tfrac{1}{2}\exp(-\xi) \times\left[\int_0^{\xi} W'(\mu)\exp(\mu)\,d\mu - \int_0^{\infty} W'(\mu)\exp(-\mu)\,d\mu\right]$$
$$= \sinh\xi\int_0^{\infty} W'(\mu)\exp(-\mu)\,d\mu - \int_0^{\xi} W'(\mu)\sinh(\xi-\mu)\,d\mu. \qquad (3.17)$$

As $\xi \to 0$, the two terms give a leading contribution, after an integration by parts,

$$\alpha^2\zeta \sim \xi\int_0^{\infty} W'(\mu)\exp(-\mu)\,d\mu - \int_0^{\xi} W(\mu)\,d\mu$$
$$\sim \xi\int_0^{\infty} W'(\mu)\exp(-\mu)\,d\mu - \tfrac{1}{3}\xi^3[\log\xi^{-1}]^{\frac{1}{3}}. \qquad (3.18)$$

The second component here is a particular integral, needed to provide the right-hand side of (3.16). However, the first term is clearly the more significant as $\xi \to 0$, and gives the form of ζ in the thermal layer, in which the appropriate approximation to (2.20) is $d^2\zeta/d\eta^2 = 0$.

3.2. *Rotational effects unimportant*: $Ta \ll O(R)$

Rotational effects do not become significant immediately Ta becomes non-zero; but they can initially be regarded as small perturbations to the previous solution. Our previous solution for ζ shows that the right-hand side of (2.19) first becomes significant in the intermediate layer when $\alpha^{-4}Ta$ becomes $O(1)$. Hence rotational effects are unimportant, so long as $Ta \ll O(R)$ for wavenumbers in the maximizing range $\alpha = O(R^{\frac{1}{4}})$.

3.3. *Development of an Ekman layer*: $O(R) \ll Ta \ll O(R\log R)^{\frac{4}{3}}$

We can now expect the vorticity derivative in (2.19) to become significant near the boundary. Provided α is still large, there can still be a uniform isothermal interior with (2.19)–(2.21) approximated, as before, by

$$\alpha^4\omega = \alpha^2\theta, \quad \alpha^2\zeta = 0, \quad \omega\theta = 1, \qquad (3.19)$$

and hence

$$\omega = 1/\alpha, \quad \theta = \alpha, \quad \zeta = 0, \qquad (3.20)$$

provided

$$O(Ta^{\frac{1}{4}}) \gg O(R^{\frac{1}{4}}) \geqslant \alpha \gg O(Ta^{\frac{1}{6}}). \qquad (3.21)$$

The first part of this inequality is required by the integral constraint (2.22), while the second is necessary to keep the vorticity derivative in (2.19), which arises from the Coriolis term, small in the interior. However, as the boundary is approached and d/dz increases in significance, we reach a stage at which Coriolis effects are as important as viscous effects from horizontal shears, so that the appropriate approximate forms of (2.19)–(2.21) are

$$\alpha^4\omega - \alpha^2\theta = Ta\frac{d\zeta}{dz}, \quad \alpha^2\zeta = \frac{d\omega}{dz}, \quad \omega\theta = 1. \qquad (3.22)$$

The scaling $\omega = W/\alpha$ is still appropriate, and the layer co-ordinate should be ξ, where now $d\xi = \alpha^3 Ta^{-\frac{1}{2}}\,dz$. (A more precise definition of ξ is given below.) Hence this new type of intermediate layer is thicker than the $O(\alpha^{-1})$ of the previous case. The governing equation for the new intermediate layer is therefore

$$\frac{d^2W}{d\xi^2} - W + \frac{1}{W} = 0, \tag{3.23}$$

and we require $W \to 1$ as $\xi \to \infty$. We also require $W \to 0$ as $\xi \to 0$, and the asymptotic forms of W and the other dependent variables as $\xi \to 0$ are readily found to be

$$\left.\begin{aligned} \omega = W/\alpha \sim (\xi/\alpha)(-2\log\xi)^{\frac{1}{2}}, \quad \theta \sim \alpha\xi^{-1}(-2\log\xi)^{-\frac{1}{2}}, \\ \zeta \sim Ta^{-\frac{1}{2}}(-2\log\xi)^{\frac{1}{2}}. \end{aligned}\right\} \tag{3.24}$$

According to Morgan, we come next to a 'thermal wind' layer of thickness τ, where $2\alpha^2\tau^2\log\tau^{-1} = 1$, which is thinner by a mere logarithmic factor than the $O(\alpha^{-1})$ layer that we had previously. The reason for having this layer would appear to be that, as $\xi \to 0$, the $d^2\zeta/dz^2$ term in (2.20) grows in relative importance, and can no longer be neglected when, in terms of the present analysis, z is $O(\tau)$ where $\tau^2\alpha^2\log(Ta\alpha^{-6}\tau^{-2})$ is $O(1)$.

The best way of discussing matters appears to be in terms of a single equation for ω, that is valid uniformly throughout all regions considered so far. We can approximate (2.19) by

$$\alpha^4\omega - \frac{\alpha^2}{\omega} = Ta\frac{d\zeta}{dz}, \tag{3.25}$$

since again $\omega\theta = 1$. Substituting with this equation for $d^2\zeta/dz^2$ in (2.20), and using (3.21), yields

$$\zeta = -\frac{1}{Ta}\frac{d}{dz}\left(\frac{1}{\omega}\right) + \left(\frac{\alpha^2}{Ta} + \frac{1}{\alpha^2}\right)\frac{d\omega}{dz} \sim \frac{d}{dz}\left(\frac{\omega}{\alpha^2} - \frac{1}{\omega Ta}\right). \tag{3.26}$$

We obtain a single equation for ω when we eliminate ζ between (3.25) and (3.26). In terms of the scaled variables $W^* = Ta^{\frac{1}{2}}\alpha^{-1}\omega$, $\xi^* = \alpha z$, it is

$$\frac{d^2}{d\xi^{*2}}\left(\frac{1}{W^*} - W^*\right) = \frac{1}{W^*} - \epsilon^* W^*, \tag{3.27}$$

where ϵ^* denotes the small quantity $\alpha^4 Ta^{-1}$. A first integral of (3.27) is

$$\left(1 + \frac{1}{W^{*2}}\right)^2\left(\frac{dW^*}{d\xi^*}\right)^2 = (\epsilon^* - 1)\log(\epsilon^* W^{*2}) + \epsilon^*(W^{*2} - 1) + \left(\frac{1}{W^{*2}} - 1\right). \tag{3.28}$$

The constant of integration is determined by the requirement that $\omega \to \alpha^{-1}$ (i.e. W^* tends to the constant value $\epsilon^{*-\frac{1}{2}}$ as $\xi \to \infty$). The previous layer is described by the scaling of (3.28), in which W^* is $O(\epsilon^{*-\frac{1}{2}})$ and $d/d\xi^*$ is $O(\epsilon^{*\frac{1}{2}})$. Now the right-hand side of (3.28), which is zero at $W^* = \epsilon^{*-\frac{1}{2}}$, is positive for all smaller values of W^*, so that W^* decreases steadily from the value $(\epsilon^*)^{-\frac{1}{2}}$ for decreasing finite ξ^*. The next stage therefore occurs when W^* is $O(1)$. The leading approximation to (3.28) is now

$$(1 + (W^*)^{-2})\frac{dW^*}{d\xi^*} = \{\log\{[(\epsilon^*)^{-1}]\}^{\frac{1}{2}} = Ta^{\frac{1}{2}}\zeta. \tag{3.29}$$

This equation is, in essence, a first integral of Morgan (1973, (3.20)), his equation for the thermal-wind boundary layer. It also follows from (3.28) that, in this region (which is of thickness τ)

$$\zeta = Ta^{-\frac{1}{2}}[\log(\epsilon^*)^{-1}]^{\frac{1}{2}}. \tag{3.30}$$

We can now see why this thermal-wind boundary layer cannot be the next layer. If it were, the vorticity ζ would be so large that it could not be brought to its boundary value $\zeta = 0$. If the layers closer to the wall are of total thickness $\delta \ll \alpha$, then (2.20) is approximately

$$\frac{d^2\zeta}{dz^2} = -\frac{d\omega}{dz},$$

whence

$$\zeta = \kappa z - \int_0^z \omega\, dz$$

for some constant of integration κ. The κz term cannot match on to (3.30), and the integral term is of magnitude $O(\delta\omega)$, which is of lesser magnitude than (3.30), since ω is $O(\alpha Ta^{-\frac{1}{2}})$. Even a layer of thickness α^{-1}, which is greater than τ, could not allow ζ to increase to the magnitude of (3.30). Hence, some other type of layer must come into play next, either one that involves higher derivatives of ω in (2.19), or one that involves the second derivative of θ in (2.21) so that $\omega\theta$ is no longer unity. It is readily seen that the former possibility happens first, and in an Ekman layer of thickness $O(Ta^{-\frac{1}{4}})$. The Ekman layer is governed by the approximate equations

$$\frac{d^4\omega}{d\phi^4} - 2\sqrt{2}Ta^{\frac{1}{4}}\frac{d\zeta}{d\phi} = 0, \quad Ta^{\frac{1}{4}}\frac{d^2\zeta}{d\phi^2} + \sqrt{2}\frac{d\omega}{d\phi} = 0, \quad \phi\sqrt{2} = Ta^{\frac{1}{4}}z. \tag{3.31}$$

(The temperature term from (2.19) is unimportant, as can be verified *a posteriori*. It is $O(\alpha^2/c^2Ta)$ relative to terms retained.)

The general solution of (3.31) that does not grow exponentially as $\theta \to \infty$ and that satisfies the boundary conditions $\zeta = \omega = d\omega/dz = 0$ at $z = \phi = 0$ is

$$\omega = c\sqrt{2} - 2c\exp(-\phi)\cos(\phi - \tfrac{1}{4}\pi), \quad Ta^{\frac{1}{4}}\zeta = 2c - 2c\exp(-\phi)\cos\phi. \tag{3.32}$$

Here c is a constant of integration, that is fixed by the requirement of matching onto the appropriate solution of (3.23) as $\xi \to 0$. It is now apparent that matching must occur at the small finite value of $\xi = \xi_0$, where

$$\alpha c\sqrt{2} = \xi_0(-2\log\xi_0)^{\frac{1}{2}}, \quad 2cTa^{\frac{1}{4}} = (-2\log\xi_0)^{\frac{1}{2}}. \tag{3.33}$$

Hence,

$$\xi_0 = \alpha/Ta^{\frac{1}{4}}\sqrt{2}, \quad c = \tfrac{1}{2}Ta^{-\frac{1}{4}}[\log(2Ta^{\frac{1}{2}}/\alpha^2)]^{\frac{1}{2}}. \tag{3.34}$$

Because of this, the definition of the variable ξ, which was fixed earlier only to within an additive constant, can now be specified more completely, and related to z by

$$\xi - \xi_0 = \xi - \alpha/Ta^{\frac{1}{4}}\sqrt{2} = z\alpha^3Ta^{-\frac{1}{2}}. \tag{3.35}$$

(Note that the intermediate layer is a boundary layer, and not an internal layer. However, the origin of ξ does not quite coincide with the physical boundary.) Moreover, matching onto the Ekman layer occurs when ω is $c\sqrt{2}$, which is larger than $O(\alpha Ta^{-\frac{1}{2}})$ in the layer of thickness τ suggested by Morgan. The present argument, therefore, is that the Ekman layer intervenes, and precludes the layer suggested by Morgan. It can be checked that the correct approximation to (2.21) in the Ekman layer is still $\omega\theta = 1$, provided that

$$(NR/Ta)\log(2Ta^{\frac{1}{2}}/\alpha^2) \gg 1. \tag{3.36}$$

The Ekman layer is unable to bring the temperature term θ to its boundary value of zero, and we must finally have a thinner thermal layer, in which conduction is important, and the approximation $\omega\theta = 1$ breaks down. The appropriate approximation to (2.21) is now

$$\frac{1}{NR}\frac{d^2\theta}{dz^2} = (\omega\theta - 1)\,\omega. \tag{3.37}$$

If we use $\delta \ll Ta^{-\frac{1}{4}}$ to denote the thickness of this layer, and let $\eta = z/\delta$ be an appropriate boundary-layer variable, then

$$\omega \sim c\phi^2\surd 2 = c\,Ta^{\frac{1}{2}}\delta^2\eta^2/\surd 2 \quad \text{and} \quad \zeta \sim c\surd 2\delta\eta$$

throughout this layer. The appropriate approximations to (2.19) and (2.20) are $d^4\omega/d\eta^4 = 0$ and $d^2\zeta/d\eta^2 = 0$, and θ is $O(1/\omega)$. So ,writing

$$\theta = \surd 2\hat{\theta}/(c\delta^2 Ta^{\frac{1}{2}}),$$

we get

$$\frac{d^2\hat{\theta}}{d\eta^2} = \frac{NRc^2Ta\delta^6}{2}\,\eta^2(\eta^2\hat{\theta} - 1). \tag{3.38}$$

The choice of δ such that

$$1 = \frac{NRc^2Ta\delta^6}{2} = \frac{NR\delta^6 Ta^{\frac{1}{2}}}{8}\left[\log\left(\frac{2Ta^{\frac{1}{2}}}{\alpha^2}\right)\right] \tag{3.39}$$

therefore gives us the same equation as (3.10) of the non-rotating case, and hence the same solution (3.11). This innermost thermal layer thus has the same basic structure as in the non-rotating case, though its thickness is altered.

Finally, to complete our solution, we must evaluate the integral constraint (2.22). The contributions from the interior and the intermediate layer and that from the thermal layer, which dominates that of the Ekman layer, give

$$R \sim \alpha^4 + \frac{4\surd 2Ta^{\frac{3}{4}}}{\log(Ta/\alpha^4)} + \frac{4I}{c^2\delta^5 Ta}. \tag{3.40}$$

Here I is the constant defined earlier (3.12), and the calculation of the intermediate-layer contribution is given in the appendix. It is much smaller than R in the parameter range under consideration, and so may be neglected. Then, using (3.34) and (3.39), we obtain

$$\left.\begin{aligned} \delta &\sim 2I^{\frac{1}{5}}[Ta^{\frac{1}{2}}R(1-\alpha^4/R)\log(4Ta/\alpha^4)]^{-\frac{1}{5}}, \\ N &\sim \tfrac{1}{4}I^{-\frac{6}{5}}R^{\frac{1}{5}}Ta^{\frac{1}{10}}(1-\alpha^4/R)^{\frac{6}{5}}[\log(4Ta/\alpha^4)]^{\frac{1}{5}}. \end{aligned}\right\} \tag{3.41}$$

It now appears that the heat flux is maximized by making α as small as possible.

The analysis of this section has supposed so far that $\alpha \gg O(Ta^{\frac{1}{6}})$; we must now consider the effect of smaller values of α. Therefore, define

$$k = Ta^{\frac{1}{2}}/\alpha^3, \tag{3.42}$$

where k may be $O(1)$ or smaller. By the time k has decreased to $O(1)$, Coriolis effects are important in the interior, and what was previously an intermediate layer has expanded to fill the interior. The interior is described by (3.23), or equivalently

$$k^2 \frac{d^2 W}{dz^2} = W - \frac{1}{W}; \tag{3.43}$$

and we need a solution for which $W \to 0$ as $z \to 0$ and 1. Its asymptotic form for small z is

$$\left.\begin{aligned} W &\sim \frac{(z+z_0)}{k}[-2\log(z+z_0)]^{\frac{1}{2}}, \\ \zeta &= \frac{1}{\alpha^3}\frac{dW}{dz} \sim \frac{1}{Ta^{\frac{1}{2}}}[-2\log(z+z_0)]^{\frac{1}{2}}, \end{aligned}\right\} \tag{3.44}$$

where, as before, the small constant z_0 is needed for matching. We can again match onto the Ekman layer (3.32) provided

$$z_0 = Ta^{\frac{1}{2}}/(\alpha^2\sqrt{2}), \quad c = \tfrac{1}{2}Ta^{-\frac{1}{4}}[\log(2\alpha^4/Ta^{\frac{1}{2}})]^{\frac{1}{2}}. \tag{3.45}$$

Note the significant change in the log term of c in (3.45) from (3.34). Provided z_0 is small, i.e. $\alpha \gg O(Ta^{\frac{1}{4}})$, we can again match on to the Ekman layer, and the subsequent thermal layer as before, with the new value of c. A recalculation of the integral constraint (2.22) is needed now that the interior is not uniform. The discussion given in the appendix shows that the contribution from the non-uniform interior is $O[Ta^{\frac{3}{4}}(\log Ta)^{-1}]$, so that it is insignificant compared with R for the parameter range under discussion. Hence the dominant contribution to the integral must be that from the thermal layer, and hence

$$R \sim 4I/(c^2\delta^5 Ta), \quad N \sim 2^{-\frac{7}{5}}I^{-\frac{2}{5}}c^{\frac{4}{5}}R^{\frac{1}{5}}Ta^{\frac{1}{5}}. \tag{3.46}$$

Consequently, N is now maximized by the largest possible value of c. Expression (3.45) shows that c increases with α in the range $O(Ta^{\frac{1}{8}}) \ll \alpha \leqslant O(Ta^{\frac{1}{4}})$. This, together with our previous result, shows that the maximum of c must occur when α is $O(Ta^{\frac{1}{4}})$ and k is $O(1)$. To find this maximum and the solution that gives the maximum heat flux, a more careful discussion of (3.43) is needed, that is valid for both large and small k.

An exact first integral of (3.43) for the interior is obtained after multiplying through by dW/dz:

$$k^2\left(\frac{dW}{dz}\right)^2 = W^2 - W_0^2 - 2\log\left(\frac{W}{W_0}\right). \tag{3.47}$$

Here $W_0 > 0$, which comes in as a constant of integration, is the value of W at the central level of $z = \frac{1}{2}$, where, by symmetry, $dW/dz = 0$. Now W_0 must be the smaller of the two roots of the right-hand side of (3.47), which must be less than one. Then $(dW/dz)^2$ has a simple zero at $W = W_0$, and it is positive for smaller W_0. The solution for W increases from the small value $\alpha c\sqrt{2}$ at $z = 0$ (the outer edge of the Ekman layer) to its maximum W_0 at $z = \frac{1}{2}$, then decreases symmetrically back to the boundary value $\alpha c\sqrt{2}$ at $z = 1$. Hence,

$$\tfrac{1}{2} = \int_{z=0}^{\frac{1}{2}} dz = \int_{\alpha c\sqrt{2}}^{W_0} \frac{dW}{dW/dz} = k\int_{\alpha c\sqrt{2}}^{W_0} \frac{dW}{[W^2 - W_0^2 - 2\log(W/W_0)]^{\frac{1}{2}}}. \tag{3.48}$$

The vorticity ζ is known explicitly in terms of W via (3.22). Applying this at the edge of the interior where $\zeta = 2cTa^{-\frac{1}{4}}$, we obtain

$$2cTa^{\frac{1}{4}} = [2\alpha^2c^2 - W_0^2 - \log(2\alpha^2c^2/W_0^2)]^{\frac{1}{2}}. \tag{3.49}$$

Equations (3.48) and (3.49) now implicitly relate the three unknown quantities α, W_0 and c. They can be simplified, if we wish to find the value of α that maximizes c, and therefore set

$$\alpha = \beta Ta^{\frac{1}{6}}, \tag{3.50}$$

where β is $O(1)$. Now k and W_0 are $O(1)$, while αc is small and $O[Ta^{-\frac{1}{12}}(\log Ta)^{\frac{1}{2}}]$. Hence, as an approximation, we can replace the lower limit of integration in (3.48) by zero, to get

$$\beta^3 = \int_0^1 \frac{2W_0\,d\mu}{[W_0^2(\mu^2-1)-2\log\mu]^{\frac{1}{2}}} + O(Ta^{-\frac{1}{12}}), \quad \mu = W/W_0. \tag{3.51}$$

We can also neglect the small α^2c^2 term in (3.51), to obtain

$$4c^2Ta^{\frac{1}{2}} + \log(2c^2Ta^{\frac{1}{3}}) = -W_0^2 + \log(W_0^2/\beta^2). \tag{3.52}$$

Now, treating both c^2 and W_0 as functions of β, we find by differentiation that, for c to be a maximum with respect to β,

$$\left(\frac{1}{W_0} - W_0\right)\frac{dW_0}{d\beta} = \frac{1}{\beta}. \tag{3.53}$$

Finally, differentiation of (3.51), and elimination of β, gives us for W_0

$$W_0^2\int_0^1 \frac{(1-\mu^2)\,d\mu}{[W_0^2(\mu^2-1)-2\log\mu]^{\frac{3}{2}}} = (2-3W_0^2)\int_0^1 \frac{d\mu}{[W_0^2(\mu^2-1)-2\log\mu]^{\frac{1}{2}}}. \tag{3.54}$$

A unique root for W_0 in the range $0 < W_0 < 1$ can be found numerically. This gives $W_0 = 0{\cdot}6857$, $\beta = 1{\cdot}2898$. The leading approximation to c can be found without reference to these numerical values. It is

$$c = (2Ta^{\frac{1}{4}})^{-1}[\tfrac{1}{6}\log Ta]^{\frac{1}{2}}, \tag{3.55}$$

in agreement with the limits of both (3.34) and (3.45); and the maximum heat flux is given by

$$N \sim \tfrac{1}{4}I^{-\frac{6}{5}}R^{\frac{1}{5}}Ta^{\frac{1}{10}}[\tfrac{1}{3}\log Ta]^{\frac{1}{5}}, \quad \alpha = 1{\cdot}2898Ta^{\frac{1}{6}}. \tag{3.56}$$

Note how weakly N depends on α in the range $O(R^{\frac{1}{4}}) \geqslant \alpha \gg O(Ta^{\frac{1}{6}})$ considered thus far.

Although the maximizing value of α changes discontinuously once Ta exceeds R, and the structure of the solution is altered markedly, (3.56) for N has the same functional dependence on R as (3.15) when $Ta \to O(R)$. An interesting qualitative feature is that the rotation acts to increase N over its non-rotating value for fixed and large values of R.

Rossby observed this phenomenon in his experiments with water (Prandtl number 6·8), but not in those with mercury (Prandtl number 0·2). His determination is that the maximum heat flux occurs at a value of R proportional to $Ta^{0{\cdot}63}$, whereas the prediction of the present single-wavenumber theory is that the maximum should occur in the extreme case for our parameter range of R of order

$(Ta)^{0.75}(\log Ta)^{-1}$ because, as we shall see in § 3.4, the Nusselt number decreases sharply when Ta is increased beyond the confines of the present parameter range. The structure derived in this section ceases to be valid when $\delta = O(Ta^{-\frac{1}{4}})$, i.e. when $Ta = O[(R\log R)^{\frac{4}{3}}]$, because by then the thickness of the thermal layer has so grown that it equals that of the Ekman layer. Similarly, approximation (3.36) has ceased to be valid.

The fact that the presence of a rotational constraint can cause an increased heat flux is somewhat surprising, in view of the stabilizing role of rotation in the linear stability problem for Bénard convection. The mechanism by which the heat flux is increased depends critically on the Ekman layer that develops when Ta exceeds $O(R)$. Once the Ekman layer has formed, the thickness of the thermal layer, which varies as N^{-1} as in any strongly convective flow, becomes thinner than its thickness when rotation is ineffectual, and temperature gradients are correspondingly intensified. Also, the effect of Coriolis force and the Ekman layer is to make the scaled vertical velocity ω larger than before. It is now

$$O(R^{-\frac{2}{5}}Ta^{\frac{1}{20}}[\log Ta]^{\frac{1}{10}})$$

in the thermal layer, rather than $O(R^{-\frac{7}{20}}[\log R]^{\frac{1}{10}})$, and $O(Ta^{-\frac{1}{6}})$ in the interior, as opposed to $O(R^{-\frac{1}{4}})$. Although Rossby is correct in suggesting that the thickness of the Ekman and thermal boundary layers are comparable when the Nusselt number is maximum, there is no mechanism by which the rigid boundary conditions are relaxed, allowing the bulk of the fluid to act as though the boundaries were free. The Ekman layer that plays a crucial role in our structure is that appropriate to a rigid boundary, rather than the weaker one that would be present with a free boundary. Our work therefore supports Veronis (1968), who, failing to find any increase of N with Ta in his numerical studies of the free-boundary case, concluded that the phenomenon does depend essentially on the rigid-boundary conditions.

To complete the discussion of this section, it should be noted that solutions of the equations are possible for values of α still smaller than those considered so far, for which

$$O(Ta^{\frac{1}{6}}) \gg \alpha \gg O[(Ta/R)^{\frac{1}{2}}]. \tag{3.57}$$

The interior now becomes uniform because of the magnitude of k. Neither ω nor W changes appreciably across the interior, so that $\omega = W_0/\alpha = \text{constant}$; the derivative of ζ is approximately constant, and

$$\zeta = \alpha^3(\tfrac{1}{2}-z)/W_0 Ta.$$

The requirement of matching onto the Ekman layer (3.32) fixes

$$W_0 = 2^{-\frac{3}{4}}\alpha^2 Ta^{-\frac{3}{8}} \quad \text{and} \quad c = 2^{-\frac{5}{4}}\alpha Ta^{-\frac{3}{8}}.$$

The integral constraint (2.22) now yields

$$R \sim 2^{\frac{3}{2}}Ta^{\frac{3}{4}} + 4I/(c^2\delta^5 Ta), \quad NRc^2 Ta\,\delta^6 = 2, \tag{3.58}$$

the contributions arising from the interior and the thermal layer. ($\alpha c/W_0$ is now no longer small.) The contribution from the interior is negligible, hence, the Nusselt number is given by

$$N \sim 2^{-\frac{19}{10}}I^{-\frac{6}{5}}\alpha^{\frac{2}{5}}R^{\frac{1}{5}}Ta^{\frac{1}{20}}, \tag{3.59}$$

so that N decreases with α through this range. Expression (3.59) also has the same functional dependence on Ta as (3.46) in the limit $\alpha \to O(Ta^{\frac{1}{6}})$. The lower limit in (3.57) is necessary for self-consistency of the solution, and for the Ekman layer to be thicker than the thermal layer.

3.4. *Thickening of the thermal layer*: $O[(R \log R)^{\frac{4}{3}}] \ll Ta \ll O(R^{\frac{3}{2}})$

With increasing Taylor number, the Ekman layer, which is needed for satisfying the dynamical boundary conditions, must eventually become the thinnest layer. Also, the thermal layer must eventually thicken as N decreases to zero with increasing Ta. Hence, we should now look for a structure in which the thermal layer matches onto the interior or the intermediate layer if there is one. The equations governing the interior and intermediate layer and the Ekman layer are as before, so that the problem essentially becomes that of determining the nature of the thermal layer.

The only solutions that appear to be possible in the present parameter range are ones in which the thermal layer is suddenly much thicker than before, and thicker rather than thinner than $O(\alpha^{-1})$. The major reason for this change comes from the integral constraint (2.22), which now prevents ω from becoming as small as it did previously before we match onto the boundary layers. Such small matching values of ω give rise to $Ta^{\frac{3}{4}}$ or $Ta^{\frac{3}{4}}(\log Ta)^{-1}$ terms in the evaluations (3.40) and (3.58) of the integral constraint, which predominate over R and cause negative heat fluxes! With a thermal layer thicker than $O(\alpha^{-1})$, we can pass from the interior or intermediate layer into a region in which ω is $O[\alpha(NR)^{-\frac{1}{2}}]$, and the heat equation (2.21) reduces to

$$\theta = \frac{NR\omega}{\alpha^2 + NR\omega^2}. \tag{3.60}$$

Supposing first that

$$O(Ta^{\frac{3}{16}}) \gg O(R^{\frac{1}{4}}) \geqslant \alpha \gg O(Ta^{\frac{1}{6}}), \tag{3.61}$$

we need to match a thermal layer of thickness δ with scaled variable $\eta = z/\delta$ on to the solution at the inner edge of the intermediate layer. Using solution (3.24) with $\xi = \alpha^3 z/Ta^{\frac{1}{2}}$, the appropriate scaled dependent variables $\hat{\omega}, \hat{\theta}, \hat{\zeta}$ are defined by

$$\left.\begin{aligned} \omega &= \frac{\alpha^2\delta\hat{\omega}}{Ta^{\frac{1}{2}}}\left[\log\left(\frac{Ta}{\alpha^6\delta^2}\right)\right]^{\frac{1}{2}}, \\ \hat{\theta} &= \frac{\omega\theta}{\hat{\omega}}, \quad \zeta = \frac{\hat{\zeta}}{Ta^{\frac{1}{2}}}\left[\log\left(\frac{Ta}{\alpha^6\delta^2}\right)\right]^{\frac{1}{2}}, \end{aligned}\right\} \tag{3.62}$$

with

$$\hat{\omega} \sim \eta, \quad \hat{\theta} \sim \eta^{-1}, \quad \hat{\zeta} \sim 1 \quad \text{as} \quad \eta \to \infty. \tag{3.63}$$

The balance of (3.60) requires

$$1 = \frac{NR\alpha^2\delta^2}{Ta}\left[\log\frac{Ta}{\alpha^6\delta^2}\right], \quad \delta \sim \frac{1}{\alpha}\left(\frac{Ta}{NR}\right)^{\frac{1}{2}}\left[\log\left(\frac{NR}{\alpha^4}\right)\right]^{-\frac{1}{2}}. \tag{3.64}$$

Equations (2.19) and (2.20) simplify, to lowest order, to

$$\frac{d\hat{\zeta}}{d\eta} = 0, \quad \frac{d\hat{\omega}}{d\eta} = \hat{\zeta}, \tag{3.65}$$

so that $$\hat{\zeta} = 1, \quad \hat{\omega} = \eta, \quad \text{and} \quad \hat{\theta} = \eta/(1+\eta^2). \tag{3.66), (3.67}$$

This solution for $\hat{\theta}$ naturally satisfies the required boundary condition on the temperature. The Ekman layer is needed to adjust the solution to satisfy the correct boundary conditions on ω and ζ. It will be given again by (3.32), and matching to the thermal layer will occur where

$$1 = \frac{\omega\sqrt{2}}{\zeta Ta^{\frac{1}{2}}} = \frac{\alpha^2\delta\eta\sqrt{2}}{Ta^{\frac{1}{2}}}, \quad c = \frac{1}{2Ta^{\frac{1}{4}}}\left[\log\left(\frac{Ta}{\alpha^6\delta^2}\right)\right]^{\frac{1}{2}}. \tag{3.68}$$

The value of η given by (3.68) must be small for self-consistency.

The principal contributions to the integral constraint (2.22) come from the interior and from the thermal layers. The latter contribution is evaluated most simply, using the first of the integrals of (2.22), as

$$NR\int(1-\omega\theta)\,dz = 2NR\delta\int_0^\infty \frac{d\eta}{1+\eta^2} = \pi NR\delta. \tag{3.69}$$

The contribution from the Ekman layer is smaller than this, because it is a thinner layer. Hence,

$$R \sim \alpha^4 + \pi RN\delta. \tag{3.70}$$

(A more careful discussion of this evaluation is given in the appendix.) Using (3.64) for δ, and writing $\alpha = \gamma R^{\frac{1}{4}}$, we can solve for N, to obtain

$$N \sim \frac{\gamma^2(1-\gamma^4)^2 R^{\frac{3}{2}}}{\pi^2 Ta}\log\left(\frac{R^{\frac{3}{2}}}{Ta}\right). \tag{3.71}$$

This expression is maximized by the choice $\gamma = 5^{-\frac{1}{4}}$. At this maximum, we have

$$\alpha \sim \left(\frac{R}{5}\right)^{\frac{1}{4}}, \quad \delta \sim \frac{5\sqrt{5}\pi Ta}{4R^{\frac{3}{2}}}\left[\log\left(\frac{R^{\frac{3}{2}}}{Ta}\right)\right]^{-1}, \quad N \sim \frac{16R^{\frac{3}{2}}}{25\sqrt{5}\pi^2 Ta}\log\left(\frac{R^{\frac{3}{2}}}{Ta}\right). \tag{3.72}$$

The Nusselt number now decreases with increasing Taylor number. The region of validity of this solution is limited at one extreme by the requirement of matching on to an interior Ekman layer as per (3.68). For $Ta^{\frac{1}{2}}/\alpha^2\delta$ to be small, it is necessary that

$$O(Ta) \gg O(R\log R)^{\frac{3}{4}}. \tag{3.73}$$

At the other extreme, N is no longer large when $O(Ta) = O(R^{\frac{3}{2}})$. However, as is well known (Chandrasekhar 1961; Niiler & Bisshopp 1965), the fluid is stable for $Ta \geqslant 2R^{\frac{3}{2}}/3\pi^2\sqrt{3}$.

The maximizing solution of this section is essentially equivalent to that given by Chan (1974) for the case of two free boundaries. Chan does not consider the matching to the interior Ekman layer, and therefore states a range of validity wider than that of (3.73) for his case. However, provided matching to an interior Ekman layer is possible, the whole structure of the solution is dominated by the heat equation, and is insensitive to the dynamical boundary conditions. Hence the distinction between free and rigid boundaries is lost. Chan's stated solution for N exceeds ours in (3.72) by a factor of 4, but this is simply due to his omission of the contribution to $\langle|\nabla\theta|^2\rangle$ from the horizontal derivatives in the thermal layer. It is also important to notice that, across the zone in which Ta increases through $O[(R\log R)^{\frac{4}{3}}]$, the thickness of the thermal layer jumps discontinuously from $O[Ta^{-\frac{1}{4}})$ to $O[(Ta^{-\frac{1}{3}}(\log Ta)^{\frac{1}{2}}]$, and there is a sharp drop in the Nusselt number from $O(Ta^{\frac{1}{4}})$ to $O[Ta^{\frac{1}{3}}(\log Ta)^{-\frac{1}{2}}]$.

Again, for completeness, we should note that solutions with less than the maximum possible heat flux are possible for values of α smaller than those in the range (3.61). We can match onto the solution (3.44) with $z_0 = 0$, provided

$$1 = \frac{NR\alpha^2\delta^2}{Ta}\,[\log(\delta^{-2})]. \tag{3.74}$$

The contribution of the thermal layer dominates the integral constraint (2.22); we obtain $1 \sim \pi N\delta$; hence,

$$N \sim \frac{\alpha^2 R}{\pi^2 Ta}\left[\log\left(\frac{\alpha^4 R^2}{Ta^2}\right)\right]. \tag{3.75}$$

This type of solution is valid for

$$O(R^{\frac{1}{4}}) \gg O(Ta^{\frac{1}{6}}) \geqslant \alpha \gg O[(Ta/R)^{\frac{1}{2}}] \gg O(Ta^{\frac{1}{3}}). \tag{3.76}$$

The lower limit arises from the requirement that N be large, so that solutions cannot be expected for any smaller values of α. Both the ranges (3.61) and (3.76) shrink to zero as stability is approached, and Ta tends to $O(R^{\frac{3}{2}})$, making $\alpha \to O(Ta^{\frac{1}{6}})$, as is found in the linear stability theory.

4. Discussion

Single-wavenumber solutions to our approximate equations of thermal convection in the presence of rotation have now been found for all except certain transition regions of the R, Ta parameter space for large R. Although our solutions have not been definitively shown to be the only ones possible, no other successful possibilities were found amongst the many tried; and the comprehensive coverage of parameter space, together with the continuous link-up with the non-rotating case at one extreme and the results of linear stability theory at the other extreme, support the idea that we have found all the possible solutions.

An important aspect of the maximizing solutions are the discontinuities that occur at the transitions between the different regions of the R, Ta parameter space. The maximizing wavenumber decreases sharply as Ta increases through $O(R)$, though the functional dependence of N on R and Ta is unaltered. The boundary-layer structure is altered at this transition with the intermediate layer expanding to fill the interior, and with the addition of an Ekman layer, but the thermal layer is not greatly altered. As Ta increases through $O[(R\log R)^{\frac{4}{3}}]$, on the other hand, the maximizing wavenumber, which has been steadily increasing with Ta prior to the transition, has a further abrupt increase. The boundary-layer structure is also altered with an abrupt increase in thickness of the thermal layer, and there is a consequent decrease in N. Although the abrupt transitions were unexpected, there does not seem to be any good reason for doubting their validity. Of course, none of the changes are necessarily discontinuous. There are presumably transition layers in R, Ta space, whose structure we have not investigated, through which the different regimes that we find can be linked.

Some of the reasons for our solutions differing so markedly from those of Morgan are described in detail in § 3.3. They involve in part the magnitude of the vorticity ζ, and we consider that the way in which we have solved explicitly for ζ and ω (rather than following others in cross-eliminating ζ between (2.19) and (2.20)) to be a significant feature of our analysis. If ζ is cross-eliminated, the rigid boundary condition $\zeta = 0$ can be converted to the form

$$\frac{d^3}{dz^3}\left(\frac{d^2}{dz^2} - 2\alpha^2\right)\omega - \alpha^2\frac{d\theta}{dz} = 0. \tag{4.1}$$

But the problem that ensues is that, when this condition is used in conjunction with a boundary-layer analysis, it may have to be satisfied to more than lowest order if ζ is to be zero. For instance, the lowest approximation to (4.1) for an Ekman layer such as described by (3.31), § 3.3, is $d^5\omega/d\phi^5 = 0$. This is indeed satisfied by rigid-boundary Ekman-layer solutions, but for the reason that $d\omega/d\phi = 0$, because $d\omega/dz = 0$ at the boundary and $d^5\omega/d\phi^5 + 4d\omega/d\phi = 0$ throughout the layer. The condition $d^5\omega/d\phi^5 = 0$ is not sufficient, however, to ensure that $\zeta = 0$, but only that the larger term $d^2\zeta/d\phi^2 = 0$. We consider that Morgan's solutions are incorrect, in part because insufficient care has been taken of the vertical vorticity, and because boundary conditions for it have not been properly satisfied.

As pointed out in § 1, our present analysis is restricted to solutions with a single-wave structure in the horizontal. Our basic equations (2.8)–(2.11) allow solutions with combinations of waves in the horizontal. The importance of such solutions was first realized by Busse (1969), who showed that they allow an upper bound for N greater than that Howard had calculated assuming a single wave, although the single-wave upper bound is in excess of experimental observations. Chan (1971) also found multi-wave solutions with any integral number of waves. The solution with two waves gives a larger upper bound for N for values of R in excess of $0{\cdot}89 \times 10^{10}$. There is a lack of experimental data at such large Rayleigh numbers, and Chan's single-wave upper bound, our (3.15), is in excess of the available data.

The essential validity of the boundary-layer methods used here and elsewhere has been tested and confirmed in the simpler problem of convection in a porous layer by Busse & Joseph (1972) and Gupta & Joseph (1973). The latter work, which is based on equations similar to ones used here, shows that the upper bound to N, found from an asymptotic analysis of a single-wavenumber solutions, is again in excess of available experimental data. Gupta & Joseph solve their equations numerically as well as asymptotically, and their numerically determined values for N are rather smaller than those obtained asymptotically. Actually, their numerical values for the upper bound of N are in striking agreement with observed heat fluxes up to $R \simeq 500$, and also follow the two-wave upper bound when it exceeds the one-wave upper bound for $R \geqslant 221{\cdot}5$. But, here too, the single-wave solution still gives a reasonable approximation to N. Hence, there is

Ta	Lower upper bound	Formula used	Observation
10^4	10·98	(3·56)	8·8
10^5	14·46	(3·56)	9·4
10^6	18·88	(3·56)	9·6
10^7	13·35	(3·72)	7·0

TABLE 1. Observed values of N compared with the lesser of formulae (3.56) and (3.72) for $R = 10^6$

considerable evidence that single-wave solutions are more effective than one has any reason to expect, despite the fact that they cannot model the plume structure that one expects for the convection cells.

A theoretically significant aspect of Chan's (1971) work is his obtaining the upper bound $N = 0{\cdot}152R^{\frac{1}{3}}$ in the asymptotic limit of both R and the number of waves going to infinity. This bound, for which there is a well-known dimensional argument, exceeds that of the single-wave value (3.15). However, multi-wave solutions may well not be so effective in increasing N for large R in problems with rotation. Our present finding is that the maximum heat flux is obtained when the thermal and Ekman layers coincide. The thickness of an Ekman layer must be $O(Ta^{-\frac{1}{4}})$, regardless of horizontal cell structure, and such an arrangement makes $N \sim Ta^{\frac{1}{4}}$. This limit is achieved in the present work as Ta tends to $O[(R\log R)^{\frac{4}{3}}]$ from below, so that $N \sim (R\log R)^{\frac{1}{3}}$. Chan's analysis of multi-wave solutions in the free-boundary case (for which he does not find it necessary to discuss the Ekman layers that are important in our work) finds that infinitely many modes are possible in the range $O(R^{\frac{4}{3}}) \leqslant Ta \leqslant O(R^{\frac{3}{2}})$ and $N \sim R^3/Ta^2$. Hence, the greatest value of N for this range occurs as Ta tends to $O(R^{\frac{4}{3}})$ from above, and is

$$N \sim R^{\frac{1}{3}} \sim Ta^{\frac{1}{4}}.$$

The Ekman layer is thinner than the thermal layer in this range, so that Chan's free-boundary solution is probably valid also for the rigid-boundary case, as it is with a single wave in § 3.4. Hence, if our present ideas are correct, and no radically different boundary-layer structure is found for the multi-wave case, there will be no order-of-magnitude increase in N from $O[(R\log R)^{\frac{1}{3}}]$.

Our single-wave upper-bound formulae for N, (3.56) and (3.72), can be compared with Rossby's observed values, as displayed in his figure 11. They are somewhat larger. Consider, for definiteness, the value $R = 10^6$, which lies in the upper range of Rossby's data. Rotation is clearly already affecting N by the time $Ta = 10^4$ and the theoretical stability limit is achieved at

$$Ta = 2R^{\frac{3}{2}}/3\pi^2\surd 3 = 0{\cdot}039R^{\frac{3}{2}} = 3{\cdot}9\times 10^7.$$

Note also the small numerical coefficient of $R^{\frac{3}{2}}$ in the preceding formula. These two features suggest that the values of Ta appropriate to our different regimes are somewhat smaller than the crudest order-of-magnitude evaluations would suggest. For $R = 10^6$, (3.56) gives the lower of the upper bounds for $Ta < 6{\cdot}3\times 10^6$ and (3.72) does so for larger values of Ta. For the values tabulated, the theoretical upper bound for N is always within a factor of two of the observed value. The numerical integrations of Somerville (1971) and Somerville & Lipps (1973) are at Rayleigh numbers in the range $1\text{–}2\times 10^4$, and Nusselt numbers in the range 2–3, which are too small for any comparison with our asymptotic theory to be worth while.

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Appendix

To a degree of approximation sufficient for the discussions of § 3.3, when $\alpha c/W_0$ is small, the integral relation (3.48) between k and W_0 may be written as

$$\frac{1}{2k} = \int_0^{W_0} \frac{dW}{[W^2 - W_0^2 - 2\log(W/W_0)]^{\frac{1}{2}}} = \int_0^1 \frac{d\mu}{[\mu^2 - 1 - (2/W_0^2)\log\mu]^{\frac{1}{2}}}. \tag{A 1}$$

It is readily seen that $1/2k$ increases steadily from zero to infinity as W_0 increases from 0 to 1. For small W_0, we can ignore the $\mu^2 - 1$ term in the denominator, and evaluate the second form of the integral, using the substitution $\mu = \exp(-s^2)$ to obtain $1/2k \sim W_0(\pi/2)^{\frac{1}{2}}$. As $W_0 \to 1$, the integral becomes large, because it has a non-integrable singularity when $W_0 = 1$. To examine this limit in more detail, write $W_0 = 1 - \epsilon$, where ϵ is small and positive. Also change the variable of integration to $\psi = 1 - \mu$. Then approximately

$$\frac{1}{2k} = \int_0^1 \frac{d\psi}{[\psi^2 - 2\psi - 2(1+2\epsilon)\log(1-\psi)]^{\frac{1}{2}}}. \tag{A 2}$$

A lowest-order approximation is obtained by expanding the integral about $\psi = 0$, where it is most significant, and retaining only the most important terms. This gives

$$\frac{1}{2k} = \int_0^1 \frac{d\psi}{[4\epsilon\psi + 2\psi^2]^{\frac{1}{2}}} + O(1) = \frac{1}{\sqrt{2}}\log(\epsilon^{-1}) + O(1). \tag{A 3}$$

Now, throughout the interior and intermediate regions for which (3.43) is valid and $O(d^2/dz^2) \ll O(\alpha^2)$, the contribution to the integral constraint (2.22) can be approximately evaluated as $\alpha^2\int dz/\omega^2$ or, using the vertical velocity as variable of integration, as before,

$$2k\alpha^4 \int_{W_e}^{W_0} \frac{dW}{W^2[W^2 - W_0 - 2\log(W/W_0)]^{\frac{1}{2}}}. \tag{A 4}$$

Here W_e represents the value of W at which we match to the adjacent boundary layer, so that $W_e = \alpha c\sqrt{2}$ for matching to the Ekman layer (3.32). We cannot replace the lower limit of integration of (A 4) by zero, because this would cause the integral to diverge. Instead we see that, if W_0 is small, the contribution to the integral from the range of integration near the lower limit is more significant than that from the remainder of the range (except perhaps that near the upper limit for W_0 close to 1). It is

$$\frac{2k\alpha^4}{W_e[-2\log W_e]^{\frac{1}{2}}}. \tag{A 5}$$

With $W_e = \alpha c\sqrt{2}$ and c given by (3.34), this term is seen to be

$$4\sqrt{2}\,Ta^{\frac{3}{4}}\,[\log(Ta/\alpha^4)]^{-1}.$$

For $W_0 \to 1$, the contribution from the range of integration near $W_0 = 1$ becomes large, as it does for the integral (A 1). Its contribution can be estimated in the same manner. It is

$$(2k\alpha^4)\frac{1}{\sqrt{2}}\log\left(\frac{1}{\epsilon}\right) \sim \alpha^4, \tag{A 6}$$

using (A 3). The expression in (3.40) is the sum of contributions from (A 6) and (A 5). The former is clearly the contribution from the interior, and the latter is that from the intermediate layer, in which ω decreases sharply.

For the estimation of the integral constraint used in § 3.4, (A 4) should not be used, in view of the fact that

$$\theta = \frac{\alpha NRW}{\alpha^4 + NRW^2} \tag{A 7}$$

and

$$k^2\frac{d^2W}{dz^2} = W - \frac{\theta}{\alpha} = W - \frac{W}{W^2 + \alpha^4/NR} \tag{A 8}$$

are now valid uniformly through the interior, intermediate layer and thermal layer. A first integral of (A 8), similar to (3.47), can be obtained. It is

$$k^2\left(\frac{dW}{dz}\right)^2 = W^2 - W_0^2 - \log\left[\frac{W^2 + \alpha^4/NR}{W_0^2 + \alpha^4/NR}\right]. \tag{A 9}$$

The contribution to the right-hand side of (2.22) is approximately

$$\int_0^1 \frac{\alpha^2\theta}{\omega}dz$$
$$= 2\alpha Ta^{\frac{1}{2}}\int_0^{W_0}\frac{dW}{(W^2+\alpha^4/NR)\{W^2 - W_0^2 - \log[(W^2+\alpha^4/NR)/(W_0^2+\alpha^4/NR)]\}^{\frac{1}{2}}}. \tag{A 10}$$

In view of the fact that α^4/NR is small, it is apparent that a major contribution to (A 10) comes from the range of integration in which W is small and $O(\alpha^2/NR)$. Its contribution may be evaluated by changing the variable of integration to $q = \alpha^{-2}W(NR)^{\frac{1}{2}}$, and evaluating the following approximation to the integral:

$$\frac{2(NRTa)^{\frac{1}{2}}}{\alpha[\log(NR/\alpha^4)]^{\frac{1}{2}}}\int_0^\infty \frac{dq}{1+q^2}. \tag{A 11}$$

Expression (A 11) agrees with (3.69). It dominates the contribution of the remainder of (A 10), except perhaps for that of the region near the upper limit when $W_0 \to 1$. The contribution of the latter may be estimated as α^4, in the same manner as before.

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